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Differential and Difference Sensitivities of Natural Frequencies and Mode Shapes of Mechanical Structures

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A method, based upon the theory of the adjoint structures, is formulated for calculating the derivatives of natural frequencies and normalized mode shapes with respect to structural parameter changes in terms of local mass, stiffness, or damping, starting with data obtained by experimental processing techniques such as modal analysis. The method applies for statically or kinematically undeterminate structures, which is not the case for most classical methods of sensitivity analysis. The method is extended to obtain large-change sensitivities and frequency response sensitivities to structural nonparameter changes (e.g., the addition of a damped vibration absorber). Two examples demonstrate the procedure and the usefulness of the sensitivity analysis.

Introduction

THE dynamic analysis of complex mechanical equipment and the prediction of the dynamic behavior of a modified mechanical structure has turned out to be a difficult way of reaching the objectives of such analysis. However, modal analysis techniques and computer-interfaced testing equipment have contributed to a solution of those problems. Furthermore, modal analysis results (complex modal displacements, natural frequencies, damping values) may be used in system synthesis methods [1,2] to predict mathematically the effect of structural changes on the dynamic behavior. The objective of this paper is to present a "sensitivity analysis" using experimental data, obtained by modal analysis, for computational assessment of the most effective parameter change in order to obtain a desired dynamic behavior.

The sensitivities give the influence of the building element parameters on the natural frequencies and mode shapes of the mechanical structures. They provide us with an answer to the question of where to change, e.g., to obtain a maximum shift of a specific natural frequency or to reduce most effectively the modal displacements in certain points for a specific mode.

The assumption of linear and statically or kinematically determinate structures simplifies the calculation of the sensitivities.[3-5] Van Belle[6,7] developed a more general method, called the theory of adjoint structures, for the sensitivity analysis of mechanical structures, yielding equations still valuable for statically or kinematically undeterminate structures.

In this paper, the method based upon the theory of adjoint structures is extended to obtain sensitivities in the case of viscously damped systems and expressions for the sensitivities are derived, using finite instead of infinitesimal changes.

Differential Sensitivities for Viscously Damped Systems

The differential equations of motion for an n degree-of-freedom viscously damped system can be expressed in matrix form as:

$$|-\omega^2 |M| + j\omega |C| + |K|| \{X\} = \{F\} \tag{1}$$

where $\{X\}$ and $\{F\}$ denote the displacement and force vectors, while $|M|$, $|K|$, and $|C|$ denote the system mass, stiffness, and damping matrices.

The steady-state response of the system, due to a sinusoidal excitation, in terms of its n complex modes may be written as:

$$\{X\} = \sum_{k=1}^{n} \left| \frac{\{\psi_k\}\{\psi_k\}^T}{a_k(j\omega-\lambda_k)} + \frac{\{\bar{\psi}_k\}\{\bar{\psi}_k\}^T}{\bar{a}_k(j\omega-\bar{\lambda}_k)} \right| \{F\} \tag{2}$$

where

$\{\psi_k\}$ = kth eigenvector
$\lambda_k = \partial_k + j\gamma_k$ eigenvalue of mode k
∂_k = exponental decay rate for mode k
γ_k = damped natural frequency of mode k
a_k = scaling factor for mode k

Normalizing the eigenvectors ($a_k = 1$, $k = 1...n$) yields:

$$\{X\} = \sum_{k=1}^{n} \frac{\{\psi_k'\}\{\psi_k'\}^T}{j\omega-\lambda_k}\{F\} + \sum_{k=1}^{n} \frac{\{\bar{\psi}_k'\}\{\bar{\psi}_k'\}^T}{j\omega-\bar{\lambda}_k}\{F\} \tag{3}$$

where

$\{\psi_k'\}$ = normalized kth eigenvector
$\{\bar{\psi}_k'\}$ = complex conjugate of $\{\psi_k'\}$
$\{\psi_k'\}^T$ = transpose of $\{\psi_k'\}$

The flexibility matrix $|S'|$ is equal to:

$$|S'| = \sum_k \frac{|S_k'|}{j\omega-\lambda_k} + \sum_k \frac{|\bar{S}_k'|}{j\omega-\bar{\lambda}_k} \tag{4}$$

where $|S_k'| = \{\psi_k'\}\{\psi_k'\}^T$.

Each submatrix $|S_{rq}'|$ of the flexibility matrix $|S'|$ can be expressed as:

$$|S_{rq}'| = \sum_k \frac{|S_{k,rq}'|}{j\omega-\lambda_k} + \sum_k \frac{|\bar{S}_{k,rq}'|}{j\omega-\bar{\lambda}_k} \tag{5}$$

The partial derivatives of the flexibility matrix $|S_{rq}'|$ to parameter P_m can be calculated in two ways:

1) Partial differentiation of Eq. (5)

$$\frac{\partial |S_{rq}'|}{\partial P_m} = \sum_k \frac{1}{(j\omega-\lambda_k)} \frac{\partial |S_{k,rq}'|}{\partial P_m} + \sum_k \frac{|S_{k,rq}'|}{(j\omega-\lambda_k)^2} \frac{\partial \lambda_k}{\partial P_m}$$
$$+ \sum_k \frac{1}{(j\omega-\bar{\lambda}_k)} \frac{\partial |\bar{S}_{k,rq}'|}{\partial P_m} + \sum_k \frac{|\overline{S_{k,rq}'}|}{(j\omega-\bar{\lambda}_k)^2} \frac{\partial \bar{\lambda}_k}{\partial P_m} \tag{6}$$

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2) Application of the theory of adjoint structures. The theory of adjoint structures is derived from the adjoint network theory, a general method for the sensitivity analysis of electrical networks. To calculate the sensitivities of the flexibility matrix $|S'|$, an additional structure with the same topology and geometry as the given structure is introduced. For linear structures the elements of the adjoint structure are defined by: $|\hat{S}| = |S'|^T$. It is now possible to prove that, for linear structures, the sensitivities of a submatrix $|S'_{rq}|$ of the flexibility matrix $|S'|$, for a parameter change P_m, are expressed by[6]:

$$\frac{\partial |S'_{rq}|}{\partial P_m} = -|S'_{r\ell}| \frac{\partial |-\omega^2 |M_{\ell\ell}| + j\omega |C_{\ell\ell}| + |K_{\ell\ell}||}{\partial P_m} |S'_{\ell q}| \tag{7}$$

$|M_{\ell\ell}|$, $|K_{\ell\ell}|$, $|C_{\ell\ell}|$ are submatrices of the mass, stiffness, and damping matrices containing those elements, influenced by the structural parameter P_m.

Substituting Eq. (5) into Eq. (7) and splitting Eq. (7) into partial fractions yields two equations (6 and 7) for $\partial |S'_{rq}|/\partial P_m$ in linearly independent terms. The equivalence of those equations for P_m influencing only $|K_{\ell\ell}|$ yields:

$$\begin{aligned}\frac{\partial |S'_{k,rq}|}{\partial P_m} = &-\frac{1}{(\lambda_k - \bar{\lambda}_k)} |S'_{k,r\ell}| \frac{\partial |K_{\ell\ell}|}{\partial P_m} \overline{|S'_{k,\ell q}|} \\ &-\frac{1}{(\lambda_k - \bar{\lambda}_k)} \overline{|S'_{k,r\ell}|} \frac{\partial |K_{\ell\ell}|}{\partial P_m} |S'_{k,\ell q}| \\ &-\sum_{s\neq k} || \frac{1}{\lambda_k - \lambda_s} || S'_{k,r\ell}| \frac{\partial |K_{\ell\ell}|}{\partial P_m} |S'_{s,\ell q}| \\ &+ |S'_{s,r\ell}| \frac{\partial |K_{\ell\ell}|}{\partial P_m} |S'_{k,\ell q}|| \\ &-\sum_{s\neq k} || \frac{1}{\lambda_k \bar{\lambda}_s} || S'_{k,r\ell}| \frac{\partial |K_{\ell\ell}|}{\partial P_m} \overline{|S'_{s,\ell q}|} \\ &+ \overline{|S'_{s,r\ell}|} \frac{\partial |K_{\ell\ell}|}{\partial P_m} |S'_{k,\ell q}||\end{aligned} \tag{8}$$

$$\frac{\partial \lambda_k}{\partial P_m} |I| = -|S'_{k,rq}|^{-1} |S'_{k,r\ell}| \frac{\partial |K_{\ell\ell}|}{\partial P_m} |S'_{k,\ell q}| \tag{9}$$

where $|I|$ = unity matrix.

Similar expressions can be derived when P_m is influencing the mass or damping matrix.

For the calculation of the derivatives of the eigenvalue to a parameter change, only the corresponding eigenvector is required. The calculation of the eigenvector derivatives does require complete dynamic information, although approximations can be made using truncated summations.

The assumption of viscous damping in the previous section has been proved necessary for a good resemblance between theoretical predictions and practical measurements of the sensitivities of mode shapes to a parameter change.

Finite Difference Sensitivities

Where to change a mechanical structure in order to obtain the most effective change in the dynamic behavior of the structure is obviously an important question. The differential sensitivities yield the influence on natural frequencies and mode shapes of an infinitesimally small change of a particular structural parameter; they are used as an indication for the most effective finite parameter change. By extending the method used in the previous section, it is possible to derive the exact expressions for finite difference sensitivities in function of the change itself.

The dynamic behavior of a viscously damped mechanical structure is described by Eq. (4); taking both summation terms of Eq. (4) together, one obtains for a submatrix of the flexibility matrix:

$$|S'_{rq}| = \sum_{k=1}^{2n} \frac{|S'_{k,rq}|}{p - \lambda_k}, \quad p = j\omega \tag{10}$$

In this notation: for $k>n$, $|S'_{k,rq}| = \overline{|S'_{k',rq}|}$; $\lambda_k = \bar{\lambda}_{k'}$, and $k' = k - n$.

The change in a structural parameter ΔP_m will affect mode shapes and eigenvalues. The difference between the new and original flexibility submatrix may be expressed as:

$$\Delta |S'_{rq}| = \sum_k \left| \frac{|S'_{k,rq}| + \Delta |S'_{k,rq}|}{p - \lambda_k - \Delta\lambda_k} - \frac{|S'_{k,rq}|}{p - \lambda_k} \right| \tag{11}$$

Dividing both terms of Eq. (11) by ΔP_m, yields:

$$\frac{\Delta |S'_{rq}|}{\Delta P_m} = \sum_k \frac{1}{1 - \dfrac{\Delta\lambda_k}{p - \lambda_k}} \left| \frac{1}{p - \lambda_k} \frac{|\Delta S'_{k,rq}|}{\Delta P_m} + \frac{|S'_{k,rq}|}{(p - \lambda_k)^2} \frac{\Delta\lambda_k}{\Delta P_m} \right| \tag{12}$$

The sensitivities of the compliance matrix for a parameter change is also given by Goddard, et al.[8]:

$$\frac{\Delta |S'_{rq}|}{\Delta P_m} = -|S'_{r\ell}| \frac{\Delta |K'_{\ell\ell}|}{\Delta P_m} ||I| + |S'_{\ell\ell}| \Delta |K'_{\ell\ell}||^{-1} |S'_{\ell q}| \tag{13}$$

where $|K'_{\ell\ell}| = -\omega^2 |M_{\ell\ell}| + j\omega |C_{\ell\ell}| + |K_{\ell\ell}|$.

Substituting Eq. (10) in Eq. (13) yields new expressions for $\Delta |S'_{rq}|/\Delta P_m$. The classical solution, which consists of splitting into partial fractions and identifying the various terms of Eqs. (12) and (13), as described before, is not applicable.

Splitting Eq. (13) into partial fractions requires the determination of the poles of

$$||I| + \left| \sum_k \frac{|S'_{k,\ell\ell}|}{p - \lambda_k} \right| \Delta |K'_{\ell\ell}||^{-1} \tag{14}$$

To avoid this difficulty, the use of the approximating Maclaurin and Taylor series seems effective. The sensitivity of the flexibility matrix for a change of a parameter may be expressed using a Taylor series:

$$\frac{\Delta |S'_{rq}|}{\Delta P_m} = \frac{\partial |S'_{rq}|}{\partial P_m} + \frac{1}{2} \frac{\partial^2 |S'_{rq}|}{\partial P_m^2} \Delta P_m + \ldots \tag{15}$$

Expanding Eq. (13) in a Maclaurin series yields:

$$\frac{\Delta |S'_{rq}|}{\Delta P_m} = -|S'_{r\ell}| \frac{\Delta |K'_{\ell\ell}|}{\Delta P_m} ||I| - |S'_{\ell\ell}| \frac{\Delta |K'_{\ell\ell}|}{\Delta P_m} \Delta P_m + \ldots || S'_{\ell q}| \tag{16}$$

Identification between Eqs. (15) and (16) yields:

$$\frac{\partial |S'_{rq}|}{\partial P_m} = -|S'_{r\ell}| \frac{\Delta |K'_{\ell\ell}|}{\Delta P_m} |S'_{\ell q}| \tag{17}$$

$$\frac{\partial^2 |S'_{rq}|}{\partial P_m^2} = 2 |S'_{r\ell}| \frac{\Delta |K'_{\ell\ell}|}{\Delta P_m} |S'_{\ell\ell}| \frac{\Delta |K'_{\ell\ell}|}{\Delta P_m} |S'_{\ell q}| \tag{18}$$

If $\Delta\,|K'_{\ell\ell}|/\Delta P_m$ is not a function of ΔP_m, one may write:

$$\frac{\Delta\,|K'_{\ell\ell}|}{\Delta P_m}=\frac{\partial\,|K'_{\ell\ell}|}{\partial P_m} \tag{19}$$

Under these conditions Eqs. (17) and (18) are replaced by:

$$\frac{\partial\,|S'_{rq}|}{\partial P_m}=-\,|S'_{r\ell}|\,\frac{\partial\,|K'_{\ell\ell}|}{\partial P_m}\,|S'_{\ell q}| \tag{20}$$

$$\frac{\partial^2\,|S'_{rq}|}{\partial P_m^2}=2\,|S'_{r\ell}|\,\frac{\partial\,|K'_{\ell\ell}|}{\partial P_m}\,|S'_{\ell\ell}|\,\frac{\partial\,|K'_{\ell\ell}|}{\partial P_m}\,|S'_{\ell q}| \tag{21}$$

A second expression for $\partial^2\,|S'_{rq}|/\partial P_m^2$ may be derived using Eq. (10):

$$\frac{\partial^2\,|S'_{rq}|}{\partial P_m^2}=\frac{\partial}{\partial P_m}\left\{\sum_k\frac{1}{p-\lambda_k}\,\frac{\partial\,|S'_{k,rq}|}{\partial P_m}+\frac{|S'_{k,rq}|}{(p-\lambda_k)^2}\,\frac{\partial\lambda_k}{\partial P_m}\right\} \tag{22}$$

Substituting Eq. (10) into Eq. (21) and splitting Eq. (21) into partial fractions yields two equations (21) and (22) for $\partial^2\,|S_{rq}|/\partial P_m^2$ in linearly independent terms. After identification one obtains expressions for $|\partial\lambda_k/\partial P_m|^2\,|I|$, $\partial^2\lambda_k/\partial P_m^2\,|I|$, and $\partial^2\,|S'_{k,rq}|/\partial P_m^2$.

For P_m influencing $|K_{\ell\ell}|$ only, and limiting the approximation to the first two terms in the Taylor series:

$$\begin{aligned}
\left|\frac{\Delta\lambda_k}{\Delta P_m}\right||I| &= \frac{\partial\lambda_k}{\partial P_m}+\frac{1}{2}\,\Delta P_m\cdot\frac{\partial^2\lambda_k}{\partial P_m^2}\\
&= |S'_{k,rq}|^{-1}\,|S'_{k,r\ell}|\,\frac{\partial\,|K'_{\ell\ell}|}{\partial P_m}\,|S'_{k,\ell q}|+\Delta P_m\Bigg\{|S'_{k,rq}|^{-1}\\
&\times\Bigg|\sum_{\substack{s\\ s\neq k}}\left|S'_{k,r\ell}\right|\frac{\partial\,|K'_{\ell\ell}|}{\partial P_m}\,\frac{|S'_{s,\ell\ell}|}{\lambda_k-\lambda_s}\,\frac{\partial\,|K'_{\ell\ell}|}{\partial P_m}\left|S'_{k,\ell q}\right|\\
&+\sum_{\substack{s\\ s\neq k}}|S'_{k,r\ell}|\,\frac{\partial\,|K'_{\ell\ell}|}{\partial P_m}\,|S'_{k,\ell\ell}|\,\frac{\partial\,|K'_{\ell\ell}|}{\partial P_m}\,\frac{|S'_{s,\ell q}|}{\lambda_k-\lambda_s}\\
&+\sum_{\substack{s\\ s\neq k}}\frac{|S'_{s,r\ell}|}{\lambda_k-\lambda_s}\,\frac{\partial\,|K'_{\ell\ell}|}{\partial P_m}\,|S'_{k,\ell\ell}|\,\frac{\partial\,|K'_{\ell\ell}|}{\partial P_m}\,|S'_{k,\ell q}|\\
&-\frac{\partial\lambda_k}{\partial P_m}\cdot\frac{\partial\,|S'_{k,rq}|}{\partial P_m}\Bigg|\Bigg\}
\end{aligned} \tag{23}$$

Similar expressions can be derived for the difference sensitivities of $|S'_{k,rq}|$, for P_m influencing $|C_{\ell\ell}|$ or $|M_{\ell\ell}|$ and for cases in which $\Delta\,|K'_{\ell\ell}|/\Delta P_m$ is a function of ΔP_m.

Frequency Response Sensitivities

It may be advantageous to calculate the effect of a structural change (characterized by its dynamic stiffness Δy) on the frequency responses in a limited frequency band at different structural points. Therefore the structural modification C, characterized by $\Delta y_{\ell\ell}$, is thought to be coupled at one point ℓ to the original structure B (Fig. 1).

The frequency responses of the coupled structure A can be written as[1]:

$$S_{ija}=S_{ijb}-\frac{S_{i\ell b}\cdot S_{\ell jb}}{S_{\ell\ell b}+|\Delta y_{\ell\ell}|^{-1}} \tag{24}$$

$$\Delta S_{ij}=S_{ija}-S_{ijb} \tag{25}$$

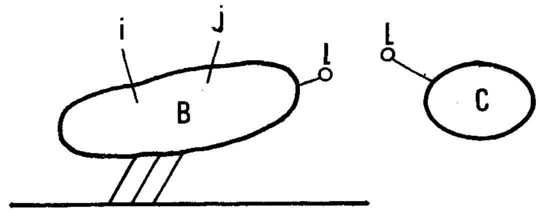

Fig. 1 Coupling of structural modification C in point ℓ, to original structure B.

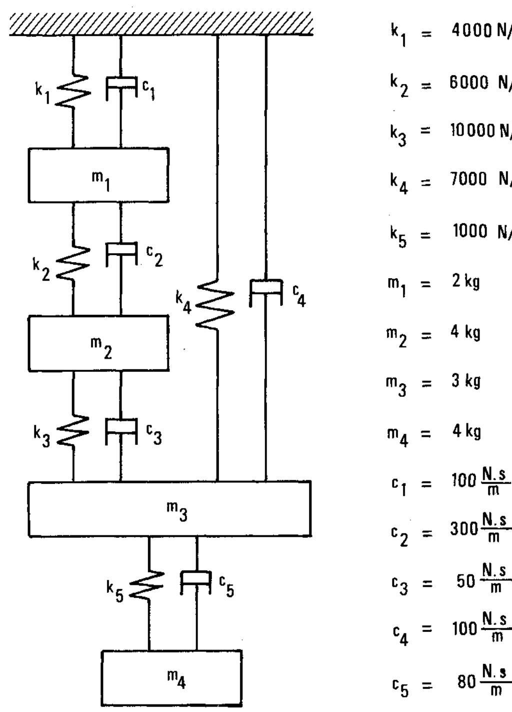

Fig. 2 Analytical example: m, k, c system with four degrees-of-freedom.

Table 1 Sensitivities of first natural frequency with respect to mass changes

Change in point	Sensitivity of first natural frequency, rad/s/kg	Computed new natural frequency after mass change of 0.1 kg, rad/s
1	0.0191	13.480
2	0.0512	13.484
3	0.0666	13.485
4	−1.39	13.341

Table 2 Sensitivities of normalized modal displacement with respect to damping change, for point 2 and mode 1

Change of damper	Sensitivity of normalized modal displacement, 10^{-6} (m/Ns)	Computed normalized displacement after damping changes of 10 N/m
1	−19.6 +16.8j	−0.013754 −0.0035221j
2	2.94 + 2.69j	−0.013562 −0.0035828j
3	1.77 − 1.59j	−0.013558 −0.0036683j
4	−14.5 +32.7j	−0.013709 −0.0033008j
5	−69 − 355j	−0.014215 −0.0082502j

where: S_{ija} = frequency response of structure A, measured in point i, with excitation in point j

$$\frac{\Delta S_{ij}}{\Delta y_{\ell\ell}}=-\frac{S_{i\ell}S_{\ell j}}{1+S_{\ell\ell}\Delta y_{\ell\ell}} \tag{26}$$

Table 3 Influence of accelerometer mass on second natural frequency of engine block

	Accelerometer type 1, $m=29.4$ g		Accelerometer type 2, $m=15.75$ g		Accelerometer type 3 $m=3.2$ g	
Point	ω_{meas}, Hz	ω_{corr}, Hz	ω_{meas}, Hz	ω_{corr}, Hz	ω_{meas}, Hz	ω_{corr}, Hz
1	1221.250	1222.499	1221.875	1222.544	1222.344	1222.480
2	1222.362	1222.399	1222.461	1222.480	1222.480	1222.484
3	1220.391	1222.292	1221.194	1222.212	1222.266	1222.474
4	1220.000	1222.440	1220.981	1222.288	1222.188	1222.453
5	1222.032	1222.424	1222.256	1222.476	1222.412	1222.465
Range	2.363	0.207	1.48	0.332	0.292	0.027
Mean		1222.421		1222.400		1222.471

Equation (26) yields the differential sensitivities:

$$\frac{\partial S_{ij}}{\delta y_{\ell\ell}} = -S_{i\ell} \cdot S_{\ell j} \tag{27}$$

Furthermore, Eqs. (26) and (27) are combined to:

$$\frac{\Delta S_{ij}}{\Delta y_{\ell\ell}} = \frac{1}{1+S_{\ell\ell}\Delta y_{\ell\ell}} \frac{\delta S_{ij}}{\delta y_{\ell\ell}} \tag{28}$$

The equations derived in this section are similar to those derived for electrical networks by Goddard et al.[8] Equation (28) can be used to predict the optimal location on the structure for an imposed structural modification. The aim of the modification can for example be to reduce as far as possible the frequency response amplitude in a given frequency band at a number of particular points of the structures.

Examples

An analytical example illustrates the theory and its possibilities. Consider the 4 degrees-of-freedom m,k,c system of Fig. 2. The first three mode shapes, damped natural frequencies, and modal damping values are computed and used in a sensitivity analysis. The influence of a mass change in the different points on the damped natural frequency of the first mode (13.478 rad/s) is given by the corresponding sensitivities. The sensitivity results are checked by recomputing the system after effectively changing the mass in the point under consideration by 0.1 kg. The results are represented in Table 1.

Out of the sensitivities, it may concluded that the first natural frequency will shift downward by reducing the mass in points 1-3, or by adding mass in point 4. Point 4 is the most sensitive point to obtain a natural frequency shift of the first mode. These conclusions are confirmed by the computational results.

A similar derivation is perfomed to reduce the normalized mode-shape displacement in the second point for the first mode (original value $-0.013576\text{-}0.0036509j$) by a damping change. From Table 2 it can be proved that the optimal solution is a reduction of damper 5 in order to achieve our goal. The results are checked by inserting 10 N/mm more damping for the damper under consideration.

In another example the influence of the accelerometer mass on the second natural frequency of an engine block is calculated for five points on the lower row on the side wall of the block. The second natural frequency and corresponding mode shape (bending) have been measured in 86 points of the engine block using the standard modal analysis technique on a H.P. 5451B Fourier Analyzer. Three accelerometers (masses, respectively, 29.4, 15.75, and 3.99 g) were used successively. The natural frequencies were calculated (using the least-squares frequency domain curve-fitting technique on frequency responses, measured using a Band Selectable Fourier Analyzer at 1216-1224 Hz). A correction for the mass of the accelerometer is applied to the experimentally determined natural frequencies. This correction was based upon the differential sensitivity expressions discussed in previous sections:

$$\omega_{corr} \simeq \omega_{meas} + \frac{\partial \omega}{\partial m_i} \Delta m_i$$

The results are represented in Table 3.

The range of the measured natural frequencies can be reduced substantially by correcting for the accelerometer mass. This correction can be important, especially for the higher natural frequencies and for heavy accelerometers. It should be emphasized that each measured frequency response should be corrected for this error before starting a curve-fitting algorithm.

Conclusions

Methods have been presented to calculate the differential and difference sensitivities of mechanical structures to parameter changes. It should be pointed out that the sensitivity expressions, based upon the theory of adjoint structures, apply for statically or kinematically undeterminate structures too. In the classical methods of sensitivity analysis, this is generally not true.

Thus emphasis is placed upon a method using experimental input data for a reduced number of points, by some modal analysis techniques. In this scope sensitivity analysis is seen as an integral part of a general technique for carrying out a dynamic vibration analysis. This analysis consists of measuring as well as processing vibrational data in order to determine the modal parameters; those modal parameters in turn are used to select the optimal spot for a modification and to determine the effect of the structural modification on the global dynamic behavior of the structure.

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